MODELING ALLOCATION OF PARALLEL FLOWS WITH GENERAL RESOURCE

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ABSTRACT
This study considers the problem of optimal allocation of a common resource in systems with parallel structures. We propose a method of solving such problems in models complicated by randomness and possible (but not required) near singularity of the constraint matrix, which commonly arises in systems with parallel processes. The method finds solution of the problem by first solving a simpler problem with an “expanded” constraint set, and then performing a directed transition to the solution of the original problem. In completing these steps the method takes account of the stochastic nature of the problem.

KEY WORDS
Parallel structure systems, modeling random flows, optimization, extension method, small parameters

1. Introduction
Allocation of physical resources or information flows in systems with parallel structures is a common practical problem. For example, distribution of a product from a warehouse to various retailers; or allocation of repair orders among various repair units, etc. (see [1-4] for other applications). Finding an optimal solution to such problems often requires costly computational procedures and methods. These high computational costs arise because the parallel structures are typically, at least partially homogeneous, which leads to ill-conditioned constraint matrices with near-singular determinants. As a result, optimization problems with such ill-conditioned constraint sets become highly unstable and hard to solve. There are methods that were developed to tackle these kinds of complications. For example, [5] studied a problem of finding stable solutions in systems with singularities in the constraint matrix by using “stabilizing functionals”. This idea has been also applied to problems with near singular constraint matrices. For instance, [6] proposes a method which first disregards small differences between nearly collinear constraints in order to extract a so called “characteristic system” of the problem, and then uses this system to assess the impact of small differences between constraints on optimal solution. The aforementioned computational methods have an important theoretical significance. However their applicability is conditional on a set of fairly stringent assumptions regarding the nature of singularity in the constraint set. Also these methods can only provide approximate solutions. An alternative solution method has been proposed in [7-11]. These studies develop an “Extension method,” for solving problems of resource allocation in systems with parallel objects with possible (but not required) near singularity of the constraint matrix. The main idea of this method is to start from a solution of a simpler problem with an expanded (i.e. relaxed) constraint set, and then perform a directed transition to the optimal solution by re-introducing the original constraints, which happen to be binding at the solution of the relaxed problem. Such differentiation between binding and non-binding constraints not only eliminates the sensitivity of the proposed method to near-singularity of the constraint set, but also allows to obtain exact solutions. This study generalizes the Extension method to a new class of resource allocation problems in systems with parallel stochastic flows.

2. Mathematical formulation of the problem
Suppose the quantity of a resource can take on any value $s$ from a set $S$, either deterministically or according to a given stochastic distribution. Further, suppose this resource is being channelled into $m$ parallel flows, where it reaches certain points (production stages) at random time points $\{t_j\}$ (Figure 1).

The problem of resource allocation for any set $T = \{t_j\}$ consists of:
maximizing a certain objective criterion

$$\max F(t_j) = f(x),$$  

(1)

subject to constraints:

$$g(x) \leq s(t_j),$$  

(2)

$$Ex = s_m(t_j),$$  

(3)

$$V \leq x \leq W,$$  

(4)

where $E$ – $n$-dimensional unit vector, $f(x)$ и $g(x)$ are assumed to be continuously differentiable and may be expressed in the following form:

$$f(x) = f_0(x) + \varepsilon f_1(x);$$

$$g(x) = g_0(x) + \varepsilon g_1(x).$$

In accordance with the extension method [7] we introduce an auxiliary extended problem, obtained from the original problem by discarding the constraints of the form (3).

$$\max F(t_j) = f(x),$$  

(5)

subject to constraints:

$$Ex = s_m(t_j),$$  

(6)

$$V \leq x \leq W.$$  

(7)

It is necessary to establish a link between solutions of the original (1)-(4) and extended (5)-(7) problems, in order to be able to solve the complex original problem by solving instead the simpler “extended” problem. The value of the objective function (5) of the extended problem at its optimal point is an upper bound on possible values of the objective function (1) of the original problem, because admissible set $X$ of the original problem is a subset of the set of admissible solutions $X^0$ of the extended problem $X \subseteq X^0$. So any transition from the point $x^0 \in X^0$ to another point $x \in X$ will worsen the objective function value. In other words, this transition will mean a descent from $F$ to a lower value of the objective function. Consider the following general algorithm of solving the problem of resource allocation among parallel flows (Figure 2).

I. Modeling elements of the discrete chain $\{t_j\}$ and the flows of the resource $s(t_j)$ and $s_m(t_j)$ in time moments $t_j$.

II. Solve the extended problem (5)-(7).

III. Verify if the obtained solution is admissible with respect to restrictions (2) of the original problem. If the decision is admissible, then it is optimal, otherwise go to step IV.

IV. Select the direction and step of a descent.

V. Transition to a new solution.

The new solution, obtained as a result of a descent would be optimal if the movement in this direction leads to the smallest change in the objective function value compared with other directions.
3. Modeling elements of the discrete chain

In modeling the elements of the discrete chain \( \{t_j\} \), let us consider a fairly general case when the set \( T = \{t_j\} \) follows a stationary Palm’s flow [12] with a given density function \( \varphi(\tau) \) over intervals between its elements, starting from the second interval. To determine the moments \( t_j \), let us use the standard formula

\[
t_j = t_{j-1} + \tau_j, j = 1, 2, \ldots, n,
\]

where \( \tau_j \) – are intervals between the elements of chain \( T \).

To model Palm’s flows it is insufficient to know \( \varphi(\tau) \), because the density function over the first interval is different from \( \varphi(\tau) \) [12], i.e.

\[
\varphi(\tau) \neq \varphi(\tau).
\]

Thus to find \( \varphi(\tau) \) we need to use Palm’s formula [12]

\[
\varphi(\tau_1) = \lambda \left( 1 - \int_0^{\tau_1} \varphi(\tau) d\tau \right), \tag{8}
\]

where \( \lambda \) – is the intensity of the flow.

The values of the intervals \( \tau_j \) between elements of the chain \( T \) are determined with the help of inverse functions of random variables according to the principles stated in the following theorem: “Realized values of the random variable \( \tau \) are determined from the formula

\[
\zeta = \int_{\tau} \varphi(\tau) d\tau = z, \quad \text{or} \quad \tau = \varphi^{-1}(z), \tag{9}
\]

where \( z \) – is a realization of the basis random variable \( \zeta \), and \( \varphi(\tau) \) is the density function” or from the discrete distribution according to the following theorem: “The values \( \tau_k \) are drawn from the table \( \begin{pmatrix} \tau_1 & \tau_2 & \ldots & \tau_m \end{pmatrix} \) and \( \begin{pmatrix} p_1 & p_2 & \ldots & p_m \end{pmatrix} \), realize with probability \( p_k \) whenever the following condition is satisfied \( z \in \Delta_k \), where \( \Delta_k = p_k \).” [12].

As the basis random variable \( \zeta \) we consider a uniformly distributed random variable on the interval [0, 1].

Proofs of the above theorems can be found in [12, 13].

If the density function \( \varphi(\tau) \) is one of the standard continuous distribution functions, then we can use the formulas provided in Table 1 for modeling the random intervals \( \tau_j \) between the elements of chain \( T \) [12].

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( f(\tau) ) density function</th>
<th>Formula for modeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma</td>
<td>( f(\tau) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\lambda\tau} )</td>
<td>( \tau = \frac{m_2 + \sigma^2}{\lambda} )</td>
</tr>
<tr>
<td>Exponential</td>
<td>( f(\tau) = e^{-\lambda \tau} ), ( \tau \geq 0 )</td>
<td>( \tau = \frac{1}{\lambda} \ln z )</td>
</tr>
<tr>
<td>Uniform</td>
<td>( f(\tau) = \frac{1}{b-a}, \tau \in [a, b] )</td>
<td>( \tau = a + \Delta )</td>
</tr>
</tbody>
</table>

A more complete list of standard continuous theoretical distributions and their formulas is provided in [13].

If, however, the intervals \( \tau \) between the elements of \( T \) take on discrete values then for modeling those values we can use one of the following formulas:

\[
\tau = \text{Int} \left[ \frac{\ln z}{\ln(1-p)} \right] + 1,
\]

\[
\tau = a + \text{Int} \left[ (b-a+1)z \right],
\]

where \( \text{Int} \left[ \right] \) stands for the integer part of the term in brackets.

4. An algorithm of the allocation of parallel flows with general resource

Let us discuss the algorithm for finding the optimal allocation of a common resource among parallel flows using a linear problem

\[
\max F(t_i) = cx_i, \quad \text{subject to} \quad Ax \leq s(t_i), \quad Ex = s_m(t_i), \quad V \leq x \leq W.
\]

Based on the general algorithm outlined above we can apply the extension method by realizing the following steps (10)-(13).

Step 1. Enter the input data

Step 2. Use Palm’s formula (8) to determine the density function \( \varphi(\tau) \).

Step 3. Using the formula (9) obtain the values

\[
\tau_i = \varphi^{-1}(z_i) \quad \text{and} \quad \tau_j = \varphi^{-1}(z_j), j > 1
\]

of the intervals \( \tau_j \) between the elements of the chain \( T \).
Step 4. Calculate the elements of the chain \( \{t_j\} \), according to \( t_j = t_{j-1} + \tau_j \).

Step 5. Using the formula (9) obtain the values
\[ s(t_j) = \Phi^{-1}(z_j) \]
of the flows of the distributed resource \( s(t_j) \) and \( s_m(t_j) \) in moments \( t_j \).

Step 6. For calculated values \( t_j \) solve the extended problem (10, 12, 13).

Step 7. Verify if the obtained solution \( x^* \) is admissible with respect to restrictions (11). If the decision is admissible, then it is optimal.

Step 8. Otherwise, in accordance with the algorithm of the extension method [7], transition to the new solution \( x = x^p + h \) and go back to step 7.

In case if the density function \( \phi(\tau) \) for the intervals between the elements of \( T \) follows one of the following commonly used laws: normal, uniform, exponential, linear, geometric, discrete uniform, Poisson and gamma distribution, then in step 3 it is possible to realize a sub-algorithm which consists of the following steps:

Step 3.1. Draw values of \( z_j \).

Step 3.2. Choose the distribution. Verify the conditions:
\[ P(\tau = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad P(\tau = m_k) = p^*(1-p)^{k-1} = p_k, \]
\[ P(\tau = k) = \frac{1}{b-a+1}, \quad f(\tau) = \frac{1}{b-a}, \]
\[ f(\tau) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\tau-m)^2}{2\sigma^2}}, \quad f(\tau) = \lambda e^{-\lambda \tau}, \]
\[ f(\tau) = \lambda \left(1 - \frac{\lambda}{2} \tau \right), \]
\[ f(\tau) = \alpha^k \frac{k (k-1) e^{-\alpha \tau}}{(k-1)!}. \]

Step 3.3. Depending on the results in step 3.2, calculate the value of \( \tau_j \) according to formulas:
\[ \tau_j = S \text{ (where } S \text{ is the number of events happening with probability } p), \quad \tau_j = \text{Int} \left[ \frac{\ln z_j}{\ln(1-p)} \right] + 1, \]
where \( \text{Int}[\ ] \) stands for the integer part of the term in brackets,
\[ \tau_j = a + \text{Int}(b - a + 1)z_j, \]
\[ \tau_j = a + m_j(b - a) + \sigma_j \left( \sum_{i=1}^{12} z_i - 6 \right), \]
\[ \tau_j = -\frac{1}{\lambda} \ln z_j, \]
\[ \tau_j = \frac{2}{\alpha} \left(1 - \sqrt{z_j} \right), \quad \tau_j = -\frac{1}{\alpha} \ln \left( z_1 z_2 \cdots z_k \right). \]

This sub-algorithm above may be also realized at the step 5 if we assume that the flows of the resource follow one of the standard theoretical distributions.

5. Numerical Example

Suppose we have a certain resource \( s \). This resource is being distributed among 4 parallel objects, with random flows measured in moments of time \( \{t_j\} \). The values of the flow are normally distributed with the density function given by
\[ \phi(\tau) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{(\tau-m)^2}{2\sigma^2}\right). \]

The intervals of time between the moments \( \tau_j \) are distributed according to the exponential law with the density function given by \( \phi(\tau) = \lambda e^{-\lambda \tau} \).

We can demonstrate our algorithm using the following linear model of resource distribution:
\[ F(x) = 5.3x_1 + 5.1x_2 + 5x_3 + 4.8x_4 \rightarrow \text{max} \]
\[ 3x_1 + 3.1x_2 + 3.2x_3 + 2.8x_4 \leq s(t_j), \]
\[ 2.1x_1 + 1.9x_2 + 2.2x_3 + 1.9x_4 \leq s_3(t_j), \]
\[ x_1 + x_2 + x_3 + x_4 = s_3(t_j), \]
\[ 1 \leq x_1 \leq 5, \]
\[ 1 \leq x_2 \leq 4, \]
\[ 1 \leq x_3 \leq 5, \]
\[ 1 \leq x_4 \leq 5. \]

1. Input the given parameters of the problem: \( t_0 = 9.00, \lambda = 0.2, m = 20.0, \sigma = 12.0 \)
2. Using Palm’s formula (8) determine the density function \( \phi(\tau_j) \):
\( \phi(t) = \lambda \left( 1 - \frac{\tau_1}{\lambda} - \ln \frac{1}{\tau_1} \right) = \lambda e^{-\lambda t} . \)

3. Since the time intervals \( \tau_j \) are assumed to follow the exponential distribution, we can use the formulas:

\[
\tau_j = -\frac{1}{\lambda} \ln z_j \quad (j = 1, \ldots, n), \quad \text{i.e.}
\]

\[
\tau_i = -\frac{1}{0.2} \ln 0.9706 = 0.1492 .
\]

4. Calculate \( t_i = t_0 + \tau_1 = 9.00 + 0.1492 = 9.1492 . \)

5. Determine the values of the flows of the distributed resource using the normal distribution. Using Marsal's algorithm [12], we obtain the following values \( s_1(t) = 30.1, s_2(t) = 20, s_3(t) = 10 . \)

6. Solve the extended problem (14), (16), (17):

\[
F(x) = 5.3x_1 + 5.1x_2 + 5x_3 + 4.8x_4 \rightarrow \max
\]

\[
x_1 + x_2 + x_3 + x_4 = 10,
\]

\[
1 \leq x_1 \leq 5,
\]

\[
1 \leq x_2 \leq 4,
\]

\[
1 \leq x_3 \leq 5,
\]

\[
1 \leq x_4 \leq 5.
\]

The optimal solution of the extended problem is \( X^* = (5; 3; 1; 1) , F^* = 51.6 . \)

7. Check if the obtained solution \( X^* \) is admissible with respect to the inequality constraints (15):

\[
3 \times 5 + 3.1 \times 3 + 3.2 \times 1 + 2.8 \times 1 = 30.1 \leq 30.1,
\]

\[
2.1 \times 5 + 1.9 \times 3 + 2.2 \times 1 + 1.9 \times 1 = 20.3 > 20 .
\]

8. The obtained solution of the extended problem is inadmissible, therefore in accordance with the algorithm of the extension method [7], we obtain a new solution \( X^* = (3.5; 3.3333; 1; 2.1667) . \) This new solution is admissible and thus, optimal. The objective value at the solution is \( F_{max} = 50.95 \).

6. **Conclusion and applications**

The results of this study were used to develop a control system for one of the largest metal producers in the world, the Ust-Kamenogorsk lead-zinc plant, which is structured into an extensive network of sequential and parallel processes [14]. The designed system controlled the sulfur acid production process, which had five different sequential production phases, taking place in dry filters, drying towers, wet filters, absorbers and contact devices. Each phase utilized 4 to 10 parallel units, with nearly identical technological parameters. Because of this homogeneity of the parallel processes, the resulting constraint matrix of the formulated model had a high degree of multi-co-linearity. As is known from [15], such equation systems have near degenerate constraint matrixes, leading to solution instability and a low degree of precision of obtained solutions. The application of the extension method with uncertain parameters allowed to obtain a solution with the error margin of less than 0.05%. Traditional solution algorithms of mathematical programming had 15% error margins and were deemed to be unacceptable by the management of the plant. Thus, practical implementation of the method shows that in contrast to other optimization methods, the proposed procedure for solving optimization problems of object placement allows to find accurate and stable solutions, even when the constraint matrix is close to being singular.

**References**


