Journal of Mathematics and System Science. - 2014. - (No. JMSS-E20141004-01)

UDC 517.5

About problems of some of the concepts of mathematical analysis

E. Zh. Aydos

erkaraai@mail.ru KazNTU after K.I.Satpayev, Kazakhstan

The article deals with the question of extending the class function where some well-known concepts of mathematical analysis, such as "continuity of the function", "smooth curve" and others are applicable. As a result of the modification of the definitions, the opportunity to use them not only for the class of differentiable functions, but also for functions with infinite derivatives has appeared.

Keywords. Limit, continuity, derivative, smooth curve, the angular function.

Introduction

It is known that the definition of concepts such as "continuity of the function", "smooth curve" and others are formulated only on the basis of differentiable functions .But to make some mathematical statement to find a wider range of applications, the notion that it is based on, in turn, should be based on a broad class of functions. Let us illustrate this on the following fact. For example, if earlier in some textbooks and manuals the mean value theorem was mainly formulated only for classes of differentiable functions (look for example [1]), but now they began to be formulated for functions that have derivatives in the broad sense (look for example . [2-3]). And therefore these theorems began to find wider application in theory and in practice. In the first case, by definition, limit or derivative of the function, which is formulated on the basis of these theorems have only valid values: $\lim\limits_{x\to x_0} f(x) = A$, ãäå $A \in (-\infty;\ +\infty)$ and $f'(x_0) \in (-\infty;\ +\infty)$, in the second case, the limit value and the derivative of the function belongs to the extended set of points on the line: $\lim\limits_{x\to x_0} f(x) \in [-\infty;\ +\infty]$ and $f'(x_0) \in [-\infty;\ +\infty]$.

Similar case takes place for the concept of **continuity of a function**. Now mathematics uses the concept of continuity, which has the following definition: a function f, defined at the point x_0 and in its neighborhood is called continuous at this point, if its limit at this point exists and $\lim\limits_{x\to x_0} f(x) = f(x_0)$. (Here $f(x_0) \in (-\infty;\ +\infty)$, otherwise the function is not defined at point x_0). According to our opinion, this definition of continuity of a function can not be considered complete. To see this, consider the definition of a smooth curve, which is based on the concept of continuity:

The curve given by the parametric equations $x = x(t),\quad y = y(t),\ t \in [a,b]$, is called smooth if $x(t),\ y(t)$ - are continuous functions with continuous derivatives on the interval $[a,b]$, simultaneously not equal to zero.

Let us try to apply this definition, for example, to a semicircle: $\Gamma_1: x=\sqrt{1-t^2}, \quad y=t, \quad -1\le t\le 1$ and to the cubic parabola: $\Gamma_2: y=\sqrt[3]{t}, \quad x=t, \ -\infty<t<+\infty$. Since the derivative of the function $x=\sqrt{1-t^2}$ is equal to $x'(t)=\begin{cases}\dfrac{-t}{\sqrt{1-t^2}}, & |t|\ne 1,\\ +\infty, & t=-1,\\ -\infty, & t=1,\end{cases}$ and the derivative of the function $y=\sqrt[3]{t}$ is equal to $y'(t)=\begin{cases}\dfrac{1}{3\sqrt[3]{t^2}}, & t\ne 0,\\ +\infty, & t=0,\end{cases}$ then both of these curves by this definition, are nonsmooth: the semicircle on the interval $[-1;\ 1]$, and the cubic parabola on the interval containing the point $t=0$.

But, this statement is absurd, since the curves that do not have corners or other specific points should be smooth (below we prove this theoretically). The reason for this absurdity is that the definition of continuity is formulated on the basis of the function that has only valid values. In this paper we show the solution to this problem by using a function with values from an expanded set of points on the line.

We think that the above problem and the method of its solution, which is given in this article, is relevant and deserves attention of mathematical science in general. We consider it necessary to introduce into the program and mathematics textbooks of high school, the definition of a derivative and continuity of the function, the definition of the smoothness of the curve, the concept of the angular function and a number of other mathematical concept, believing that the value of the function and its derivative may belong to an extended set of points $[-\infty;\ +\infty]$. The primary goalof the author is to bring outlined questions in this paper, if possible, to the wider public of mathematicians, because he hopes that this method for the determination of some notion of mathematical analysis will be one of the steps of the constructionof trouble free theory of mathematics.

The concept of continuity in the broad sense and its application

We say that *the function* f *is defined in the broad sense*, if $f(x)\in[-\infty;+\infty]$, $x\in\Delta$. In other words, function, defined in the broad sense in an Δ interval, may take finite or infinite values at the points of this interval. For example, the function $f(x)=\begin{cases}\dfrac{1}{x-3}, & x\ne 3,\\ +\infty, & x=3\end{cases}$ is *defined in the broad sense in the* $(-\infty;+\infty)$ *interval*, but $f(x)=\dfrac{1}{x-3}$ is not defined at the point $x=3$.

Definition 1 (continuity in the broad sense). Let the function f *defined* at the point x_0 and in some neighborhood *in the broad sense*. Then, if there is a limit of the function f, when

$x \to x_0$ and $\lim_{x \to x_0} f(x) = f(x_0)$, is held then the function f is called *continuous in the broad sense* at this point.

For example, the function $f(x) = \begin{cases} \frac{1}{x-5}, & x \neq 5, \\ +\infty, & x = 5 \end{cases}$ is continuous in the broad sense at the point $x = 5$, as $\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{1}{(x-5)^2} = +\infty = f(5)$.

The function, which is continuous in the broad sense in the given interval, may be continuous or continuous in the broad sense at the points of this interval.

Let us define a smooth curve in the language of a continuous derivative in the broad sense.

Definition 2. The curve $\Gamma \subset R^3$ given by parametric equations $x = x(t),\ y = y(t),\ z = z(t),\ t \in [a; b]$is called smooth, if $x(t),\ y(t),\ z(t)$ - continuous functions with ***continuous in the broad sense derivatives*** in$[a; b]$ simultaneously not equal to zero.

Definition 2 can be used to the curves represented by differentiable functions in the considered interval, as well as to the curves, represented by the functions with infinite derivatives. A special case of the definition 2 is the following

Definition 3. The curve $\Gamma : y = f(x),\ x \in [a; b]$is called*smooth*, if the function f has continuous in the broad sense derivative in the interval $[a; b]$.

These definitions can be applied to the curves represented by differentiable functions on the interval under consideration , as well as to the curves represented by functions with infinite derivatives. For example, according to the formulated definition 2, the semicircle Γ_1 and cubic parabola Γ_2, which are defined by the parametric equations above, are smooth curves, since the derivatives $x'(t)$ and $y'(t)$ are continuous in the broad sense on the considered intervals.

Further, suppose that the function $y = f(x),\ x \in [a; b]$ is given and let its derivative exist in $[a; b]$. The function

$$\alpha(x) = arctg\ f'(x), \quad x \in [a; b], \qquad (1)$$

where $f'(x) \in [-\infty; +\infty]$, is said to be *an angular function* for f. Here the function $arctgv$ considered to be defined on the extended set of real numbers$[-\infty; +\infty]$, with the conditions of $arctg(+\infty) = \frac{\pi}{2}$ and $arctg(-\infty) = -\frac{\pi}{2}$, i.e $-\frac{\pi}{2} \leq \alpha(x) \leq \frac{\pi}{2}$. Defined this way function $arctgv$ is continuous in $[-\infty; +\infty]$.

From equation (1) it is seen that the existence of the derivative $f'(x)$ is equivalent to the existance of an angular funcition at the point *x*. Similarly, the continuity of the derivative $f'(x)$ in the broad sense is equivalent to continuity in the usual sense of the angle funciton *α*(*x*).

Indeed, suppose that derivative f' at the point x_0 is continuous in the broad sense, i.e. $\lim_{x\to x_0} f'(x) = f'(x_0)$, where $f'(x)\in[-\infty;+\infty]$, . Then, if $f'(x)\in(-\infty;+\infty)$, the angular function*α*(*x*) is continuous at the point x_0, as a composition of two continuous functions. If, for example, $f'(x_0)=+\infty$, then $\alpha(x_0)=arctg(+\infty)=\frac{\pi}{2}$, and by the continuity of function $arctgv$ in $[-\infty;+\infty]$ we get $\lim_{x\to x_0}\alpha(x)=\lim_{x\to x_0} arctgf'(x)=arctg\lim_{x\to x_0} f'(x)=arctgf'(x_0)=\frac{\pi}{2}=\alpha(x_0)$.

Conversely, suppose the angular function α is continuous at the point x_0 i.e $\lim_{x\to x_0}\alpha(x)=\alpha(x_0)$, where $\alpha(x_0)=arctgf'(x_0)$. Then,taking into consideration inequalities

$$-\frac{\pi}{2}\le\alpha(x)\le\frac{\pi}{2},$$

$\forall x\in[a;b]$, we have $\lim_{x\to x_0} f'(x)=\lim_{x\to x_0} tg\alpha(x)=tg\lim_{x\to x_0}\alpha(x)=tg\alpha(x_0)=tgarctgf'(x_0)=f'(x_0)$. Here it means that, if, for example $\alpha(x_0)=arctg\ f'(x_0)=\frac{\pi}{2}$, i.e $f'(x_0)=+\infty$, then $\lim_{x\to x_0}\alpha(x)=\frac{\pi}{2}-0$, and so $\lim_{x\to x_0} f'(x)=\lim_{x\to x_0} tg\alpha(x)=\lim_{x\to x_0}\sin\alpha(x)\cdot\lim_{x\to x_0}\frac{1}{\cos\alpha(x)}=+\infty=f'(x_0)$. Thus, according to the continuity of the angular function α at the given point , continuity of the derivative f' at this point in the broad sense is followed. Therefore Definition 3 is equivalent to following definition 4, formulated in *terms of the angular funcitons*.

Definition 4. (in terms of the angular function) The continuous curve $\Gamma: y=f(x),\ x\in[a;b]$, is called *smooth*, if f has an angular function, continuous in $[a,b]$.

For example, considered above curve $y=\sqrt[3]{x},\ -\infty<x<+\infty$, is smooth according to the definition 4,too. Indeed, for $f(x)=\sqrt[3]{x}$, which is continuous at any point, angular function

$$\alpha(x)=\begin{cases} arctg\dfrac{1}{\sqrt[3]{x^2}}, & x\ne 0,\\ \dfrac{\pi}{2}, & x=0,\end{cases}$$ is defined.

For the function $f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\ne 0,\\ 0, & x=0\end{cases}$ the angular function equal to

$$\alpha(x)=\begin{cases} arctg\left(2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}\right), & x\ne 0,\\ 0, & x=0,\end{cases}$$ is defined, but it is not continuous at the point $x=0$ (

the function $\cos\frac{1}{x}$ has no limit at $x \to 0$), so the curve, given by the function f, , is not smooth in sense of the definition 2.3 in the interval, which has the point $x = 0$.

Let $A(x;\ f(x))$ to be some point of continuous curve Γ: $y=f(x)$, $x \in [a,b]$. We take the point $(x+h) \in (a,b)$ and choose the direction of the line S, which passes through the point $A(x;\ f(x))$ and $B(x+h;\ f(x+h))$, so that the angle β between the positive direction of the Ox axis and direction of the line S would be sharp: $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. We call directed line S,which is obtained in such a way, the secant, and $\beta = \beta(x;h)$ its angle of inclination.

Definition 5. If there is a limit $\lim_{h\to 0}\beta(x,h)$, then directed line T (limiting position of directional secant), passing through the point $A(x,\ f(x))$ with the angle of inclination $\alpha(x) = \lim_{h\to 0}\beta(x,h)$, is called *tangent* to the curve at this point.

Note that the angle of inclination of the secant is in the range $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$, while the slope of the tangent can take values from the interval $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$.

Thus, $\alpha(x) = \lim_{h\to 0}\beta(x,h) = arctg\ f'(x)$ is the slope of the tangent to the curve at the point with abscissa *x*. Consequently, the existence of the derivative of the function *f* at the point *x* implies the existence of a tangent to the curve at the point with abscissa *x*. However, the fact that a tangent exists at the point of the curve with abscissa *x* is possible, but the function, which defines this curve is not differentiable at the point *x*.

Further, let the angular function $\alpha = \alpha(x)$ to be defined for the function *f* at the point *x* and some neighborhood. Then the tangent at the point $(x; f(x))$ of the graph of the function *f* is continuous at the point *x*, if the angular function α is continuous at this point. We say that the tangent of the graph of the function *f* is continuous in the interval Δ, if it is continuous at each point of the Δ.

Definition 6. (in terms of the tangent) The continuous curve $\Gamma:\ y = f(x),\ x \in [a;b]$ is called *smooth*, if there is a continuous tangent of the curve Γ in the interval $[a;b]$.

References

1 G.M.Fikhtengol'ts , Course of differential and integral calculus, Vol.1,"Science", 1966.

2L.D.Kudryavtsev, Course of mathematical analysis, Volume 1, "High School", 1981.

3 E.Zh.Aydos, ZhogaryMatematika -2, "Bastau" 2014.